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SYSTEMS OF CONSERVATION LAWS

by

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ABSTRACT

In this paper a wide class of difference equations is described for approximating discontinuous time dependent solutions, with prescribed initial data, of hyperbolic systems of nonlinear conservation laws. Among these schemes we determine the best ones, i.e., those which have the smallest truncation error and in which the discontinuities are confined to a narrow band of 2-3 meshpoints. These schemes are tested for stability and are found to be stable under a mild strengthening of the Courant-Friedrichs-Lewy criterion. Test calculations of one-dimensional flows of compressible fluids with shocks, rarefaction waves and contact discontinuities show excellent agreement with exact solutions. In particular, when Lagrange coordinates are used, there is no smearing of interfaces.

The additional terms introduced into the difference scheme for the purpose of keeping the shock transition narrow are similar to, although not identical with, the artificial viscosity terms, and the like of them introduced by Richtmyer and von Neumann and elaborated by other workers in this field.

1. DIFFERENCE SCHEMES FOR CONSERVATION LAWS

In this paper we consider systems of conservation laws, i.e., equations of the form

$$u_t = f_x, \tag{1.1}$$

where u is an unknown vector function of x and t with n components, and f a given vector function of u, depending in general non-linearly on u. When the differentiation on the right side of (1.1) is carried out a quasi-linear system results:

$$u_t = A\, u_x, \tag{1.1'}$$

where $A = A(u)$ is the gradient of f. We require that our system of equations be hyperbolic in the sense that the matrix A has n distinct real eigenvalues $\mu_1, \ldots, \mu_n$, for all values of u. Of course the eigenvalues themselves are functions of u.

The negatives of the quantities μ are the local sound speeds. We shall index them in, say, monotonic increasing order.

The initial value problem is to find a solution of (1.1) with prescribed values at $t = 0$:

$$u(x,0) = \varphi(x). \tag{1.2}$$

It is well known that on account of the nonlinearity of (1.1) no smooth solution will in general exist for all time. Instead we have to seek weak solutions of (1.1), (1.2), defined by the requirement that the integral relation

$$\int\int (w_t\, u - w_x\, f)\, dxdt + \int w(x,0)\, \varphi(x)\, dx = 0 \tag{1.3}$$

be satisfied for all smooth test vectors w which vanish for $|x| + t$ large enough.

An immediate consequence of the integral relations is that every piecewise continuous weak solution must satisfy the Rankine-Hugoniot relation across a line of discontinuity:

$$s[u] + [f] = 0; \tag{1.4}$$

here the brackets denote the jump across the discontinuity, s the speed of propagation of the discontinuity.

It is well known in the theory of systems of conservation laws (see, e.g., section 7 of [9]) that in addition one has to impose a so-

called entropy condition on discontinuities. This condition can be formulated as follows:

There is an index k such that the shock speed lies between the $(k - 1)$ and kth characteristic speeds with respect to the state on the left of the shock, and between the kth and $(k + 1)$ characteristic speeds on the right.

We shall describe now a fairly large class of difference schemes that can be used to approximate weak solutions with prescribed initial data.

First we choose a vector valued function g of 2ℓ vector arguments, related to f by the sole requirement that when all the 2ℓ arguments are equal, g reduces to f:

$$g(u, \ldots, u) = f(u). \tag{1.5}$$

Abbreviate the values of $u(x)$ at the lattice points $x + k\Delta x$ and at time t by u_k, $k = -\infty, \ldots \infty$. We define

$$g(x + \Delta x/2) = g(u_{-\ell+1}, u_{-\ell+2}, \ldots, u_\ell);$$

therefore similarly we have

$$g(x - \Delta x/2) = g(u_{-\ell}, u_{-\ell+1}, \ldots, u_{\ell-1}).$$

We take now the following difference analogue of (1.1):

$$\frac{\Delta u}{\Delta t} = \frac{\Delta g}{\Delta x}, \tag{1.6}$$

where Δu denotes the forward time difference

$$\Delta u = u(x, t+\Delta t) - u(x,t)$$

and Δg the symmetric space difference

$$\Delta g = g(x + \Delta x/2) - g(x - \Delta x/2).$$

From (1.6) we can determine $u(x, t+\Delta t)$:

$$u(x, t+\Delta t) = u(x,t) + \lambda \Delta g, \tag{1.6'}$$

where λ abbreviates the quotient $\Delta t/\Delta x$ of the time and space increments. Given the initial values φ of u, we can, using (1.6'), determine successively the values of u at all times which are integer multiples of Δt. We claim now that as consequence of (1.5) our difference scheme is consistent with the differential equation in the following sense: denote by $v(x,t)$ this solution of the difference equation (which for noninteger multiples t of Δt is defined, for the sake of convenience, as equal to $v(x,t')$, $t' = \Delta t[t/\Delta t]$). This v depends, of course, on Δt and Δx.

Theorem: Assume that as Δx and Δt tend to zero, $v(x,t)$ converges boundedly almost everywhere to some function $u(x,t)$. Then $u(x,t)$ is a weak solution of (1.1) with initial values φ.

Proof: Multiply Eq. (1.6'), satisfied by v, by any test vector w, integrate with respect to x and sum over all values of t which are integer multiples of Δt. Apply summation by parts to the left side, and transform the terms involving Δg by replacing the variable of integration x by $x - \Delta x/2$ and $x + \Delta x/2$, respectively. We get the following identity:

$$\sum \int \frac{w(x,t - \Delta t) - w(x,t)}{\Delta t} v(x,t)\, dx\, \Delta t \qquad (1.3')$$

$$- \int w(x,0)\, \varphi(x)\, dx$$

$$= -\sum \int \frac{w(x + \Delta x/2) - w(x - \Delta x/2)}{\Delta x} g\, dx\, \Delta t;$$

g denotes $g(v_1, \ldots, v_{2\ell})$, where $v_1, \ldots, v_{2\ell}$ denote values of v at 2ℓ points at distance Δx apart distributed symmetrically around x,t. If v tends boundedly almost everywhere to $u(x,t)$, so do $v_1, \ldots, v_{2\ell}$, and thus $g(v_1, \ldots, v_{2\ell})$ tends to $g(u, \ldots, u)$, which, by the consistency requirement (1.5), is $f(u)$. So the limit of (1.3') is the desired integral relation (1.3).

2. MINIMIZING THE TRUNCATION ERROR

Let $u(x,t)$ be an exact smooth solution of equation (1.1). It will then satisfy the difference Eq. (1.6') only approximately; the deviation of the right side from the left side of (1.6') is called the truncation error. It is easy to see that if the function g satisfies the consistency condition (1.5), the truncation error is at least $0(\Delta^2)$. In this section we shall determine g so that the truncation error is of as high an order as possible. Specifically we shall consider the case $\ell = 1$, and show that g can be chosen so that the truncation error is $0(\Delta^3)$:

Expand $u(x, t+\Delta t)$ into a Taylor series up to terms of second order:

$$u(x, t+\Delta t) = u(x,t) + \Delta t\, u_t + \frac{(\Delta t)^2}{2} u_{tt} + 0(\Delta^3). \tag{2.1}$$

With the help of the differential equation (1.1) which u is supposed to satisfy we can express the time derivatives of u as space derivatives:

$$\begin{aligned} u_t &= f_x, \\ u_{tt} &= f_{xt} = f_{tx} = (Au_t)_x = (A^2 u_x)_x; \end{aligned} \tag{2.2}$$

what is significant is that all t derivatives are exact x derivatives and therefore can be approximated by exact x differences. Substitute, namely, (2.2) into (2.1); we get

$$u(x,\ t+\Delta t) \ = \ u + (\Delta t f + \frac{(\Delta t)^2}{2} A^2 u_x)_x + O(\Delta^3). \tag{2.3}$$

Comparing (2.3) with (1.6'), we see that the truncation error is $O(\Delta^3)$ if and only if

$$\frac{\Delta g}{\Delta x} \ = \ (f + \frac{\Delta t}{2} A^2 u_x)_x + O(\Delta^2).$$

From this we can easily determine the form that g has to take:

Theorem: The truncation error in the difference scheme (1.6') is $O(\Delta^3)$ if and only if

$$g(a,b) \ = \ \frac{f(a) + f(b)}{2} + \frac{\lambda}{2} A^2 \ . \ (b - a) \tag{2.4}$$

plus terms which are $O(|a - b|^2)$ for $u - v$ small.

Formula (2.4) has a fairly intuitive meaning and can be derived by differencing our differential equation (1.1) as follows: replace time and space derivatives by differences centered at $x,\ t + \Delta t/2$. This means that u_t is to be replaced by a forward time difference, and f_x by

$$\Big\{f(x + \Delta x/2,\ t + \Delta t/2) \ - \ f(x - \Delta x/2,\ t + \Delta t/2)\Big\}/\Delta x.$$

The value of f at $x + \Delta x/2,\ t + \Delta t/2$ is evaluated on the basis of the formula

$$f(x \pm \Delta x/2,\ t + \Delta t/2) = f(x \pm \Delta x/2,\ t) + \frac{\Delta t}{2} f_t + O(\Delta)^2.$$

We express f_t as $Au_t = A^2 u_x$, and approximate u_x by a difference quotient. The value of $f(x \pm \Delta x/2,\ t)$ is evaluated as the average of f at x and $x \pm \Delta x$. The resulting formula is precisely (2.4).

The quantity A^2 in (2.4) shall be taken as $\left\{A^2(u) + A^2(v)\right\}/2$, for sake of symmetry more than anything else; any other choice for it would make a difference that is quadratic in $u - v$.

Denoting the function (2.4) by g_o, we can write any permissible g in the form

$$g = g_o + \frac{1}{2} Q(a,\ b) \cdot (b - a), \tag{2.5}$$

where $Q(a,b)$ is a matrix which vanishes when its two vector arguments are equal.

Substituting formula (2.5) for g into the difference Eq. (1.6'), we get

$$u(x,\ t+\Delta t) = u(x,t) + \lambda \Delta' f + \frac{\lambda^2}{2} \Delta A^2 \Delta u + \frac{\lambda}{2} \Delta Q \Delta u, \tag{2.6}$$

where Δ' denotes the operation $\frac{1}{2}[T(\Delta x) - T(-\Delta x)]$, Δ the operator

$T(\Delta x/2) - T(-\Delta x/2)$. $T(s)$ denotes translation of the independent variable by the amount s. We shall call Q the artificial viscosity, since it occurs in the difference equations in a way which is similar to the artificial viscosity terms introduced by von Neumann and Richtmyer in [13].

The difference Eq. (2.6) expresses the value of u at time $t + \Delta t$ as a nonlinear function of u at t; we shall denote this function (operation) by N:

$$u(t + \Delta t) = N\, u(t). \tag{2.6'}$$

The value of the solution of the difference equation at some later time $k\Delta t$ is obtained from the initial values by application of the kth power of the operator N.

Our aim is to show that the difference scheme (2.6) is convergent if the size of λ is suitably restricted. Now in the case of linear equations it is well known and easy to show (see, e.g., [10]) that convergence is equivalent to stability defined as the uniform boundedness of all powers N^k of the operator N within some fixed range $p\Delta t \leq T$. In the nonlinear case von Neumann has made the reasonable assumption, (see, e.g., [12]) that the convergence of the scheme would hinge on the stability of the first variation of the operator N. The first variation of N is a linear difference operator with variable coefficients; von Neumann has conjectured that such an operation is stable if and only if all the "localized" operators associated with it, i.e., the operators

obtained by replacing the variable coefficients by their value at some given point -- are stable.[1]

The stability of a difference operator with constant coefficients is easily ascertained by making use of the Fourier transform. In that representation the application of a difference operator with constant coefficients goes over into multiplication by an "amplification matrix." In case the amplification matrix has n linearly independent eigenvectors, the operator is stable if and only if the eigenvalues of the amplification matrix do not exceed $1 + 0(\Delta t)$ in absolute value.

On the other hand, Courant, Friedrichs and Lewy have observed in their classical paper [2] that a necessary condition for the convergence of a difference scheme is that the rate of propagation of signals in the difference scheme should be at least as large as the true maximum signal speed, i.e.,

$$\frac{\Delta x}{\Delta t} \geq |\mu|_{max}, \tag{2.7}$$

where $|\mu|_{max}$ is the largest eigenvalue of A at any point within the relevant range of values.

Conversely, we shall show:

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[1] In the parabolic case the validity of this conjecture has been established by Fritz John in his important paper [4]. In the hyperbolic case a fragmentary result has be given in Lax [8].

Theorem: If condition (2.7) is fulfilled, the difference Eq. (2.6) satisfies von Neumann's condition of stability.

Proof: The first variation of the operator appearing on the right side of (2.6) is easily computed; its value is

$$I + \lambda A\Delta' + \frac{1}{2}\lambda^2 A^2\Delta^2 + 0(\Delta x); \tag{2.8}$$

where Δ' and Δ are defined as before, and $0(\Delta x)$ denotes an operator bounded in norm by $0(\Delta x)$, provided that we are perturbing in the neighborhood of a smoothly varying solution, i.e., one where neighboring values differ by $0(\Delta x)$. In this case the influence of the additional artificial viscosity term is $0(\Delta x)$; in section 3 we shall show how to take into account the effect on stability of the Q term in regions where u does vary rapidly.

To "localize" the operator (2.8) we merely replace the variable matrix A by its value at some point. After making a Fourier transformation, the operator Δ' becomes multiplication by $i \sin \alpha$, and the operator Δ multiplication by $2i \sin \frac{\alpha}{2}$, so that $\frac{1}{2}\Delta^2$ becomes multiplication by $\cos \alpha - 1$; here α is $\xi\Delta x$, ξ the dual variable. So the amplication matrix of the operator (2.8) is

$$I + i \sin \alpha\, \lambda A + (\cos \alpha - 1)\, \lambda^2 A^2 + 0(\Delta x). \tag{2.9}$$

Denote the eigenvalues of λA by k; then, according to the spectral

mapping theorem, the eigenvalues v of the amplification matrix (2.9) are

$$v = 1 + i\,k \sin\alpha + k^2(\cos\alpha - 1) + 0(\Delta x).$$

Since $k = \lambda\mu$ is real, the absolute value of v is given by

$$|v|^2 = (1 - [1 - \cos\xi]\,k^2)^2 + k^2 \sin^2\xi + 0(\Delta x)$$

$$= 1 - 2[1 - \cos\xi]\,k^2 + [1 - \cos\xi]^2\,k^4 + k^2(1 - \cos^2\xi) + 0(\Delta x)$$

$$= 1 - [1 - \cos\xi]^2\,(k^2 - k^4) + 0(\Delta x).$$

According to our basic stability assumption (2.7) the quantity k does not exceed 1 in absolute value; the above formula for $|v|$ shows that $|v|$ is bounded by $1 + 0(\Delta x)$. Thus we have demonstrated that the eigenvalues of the amplification matrix do not exceed $1 + 0(\Delta x)$ in absolute value, as required in von Neumann's condition.

3. ARTIFICIAL VISCOSITY FOR SINGLE CONSERVATION LAWS

In the last section we have determined the function $g(u,v)$ up to quadratic terms in $u - v$. These undetermined quadratic terms influence neither the order of the truncation error nor the stability of the scheme at points where the solution varies smoothly. At points, however, where the solution varies rapidly -- across a shock, that is -- it is reasonable to expect that the quadratic terms have a controlling influence. In this section we shall show how to choose the additional quadratic term Q in our formula (2.6) so as to assure a fairly narrow shock width.

As observed in the last section, Q enters the difference Eq. (2.6) like an artificial viscosity. Therefore, in order to insure that Q has a stabilizing influence on the difference equation, it is reasonable to require that: a. Q is positive. Another property of Q, already previously noted, is: b. $Q(a,b) = 0$ when $a = b$, while on dimensional grounds we must require that: c. Q has the dimension of A. We shall discuss first the scalar case, i.e., when the number of components of u (and f) is one; correspondingly the matrix $A = \text{grad } f$ is one-by-one. In this case the restrictions a - c show that Q must be a multiple of $|A(a) - A(b)|$:

$$Q(a,b) = \frac{B}{2} |A(a) - A(b)|, \tag{3.1}$$

where B is a dimensionless constant (which of course could depend on the value of u and v). With the above choice for Q in the difference equation (3.1) we shall now study the shape of a steady state shock, i.e., a time independent solution of (2.6) connecting states $\mu(-\infty)$ and $\mu(\infty)$:

$$\Delta' f + \frac{\lambda}{2} \Delta A^2 \Delta u + \frac{1}{2} \Delta Q \Delta u = 0. \tag{3.2}$$

We shall denote the values of u at three successive lattice points by a, b and c. We write,

$$\Delta' f = \frac{1}{2}[f(c) - f(b)] + \frac{1}{2}[f(b) - f(a)]$$

and make the following approximation:

$$f(c) - f(b) \approx A(b,c)(c-b),$$

$$f(b) - f(a) \approx A(a,b)(b-a),$$

where $A(u,v)$ abbreviates $\frac{A(u) + A(v)}{2}$. For the simplest nonlinear function f, a quadratic one, these approximations involve no error.

Substituting the above changes into (3.2), we get, after multiplication by 2:

$$\left\{A + \lambda A^2 + Q\right\}(c-b) + \left\{A - \lambda A^2 - Q\right\}(b-a) = 0. \qquad (3.3)$$

We make now a further approximation by omitting the terms λA^2 in (3.3), under the reasonable assumption that the presence or absence of this term won't alter much the nature of the solution, since this term has the same character as the retained Q terms, and is expected to be much smaller than the Q term in the important region of rapid variation, especially if λ is small.

This assumption was put to the following numerical test: in the difference equation (2.6), A^2 was replaced by WA^2, where $W(a,b)$ is a factor so contrived that its value is near 1 when u and v are close, while its value is near zero when u and v differ greatly.[2] The presence or absence of such a factor W made hardly any difference in the shape of the steady state shock profile obtained experimentally.

Making this change, and substituting our choice (3.1) for Q into (3.3), we get

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[2] I.e., W switches off the higher order correction term in the shock zone, where it has no function to fulfill anyway.

$$\left\{A(b,c) + \frac{B}{2}\,|A(c) - A(b)|\right\}(c-b) + \left\{A(a,b) - \frac{B}{2}\,|A(b) - A(a)|\right\}(b-a) = 0.$$

When $B = 1$, this equation can be written in the following simple form:

$$\text{Max}\left\{A(b),\ A(c)\right\}(c-b) + \text{Min}\left\{A(a),\ A(b)\right\}(b-a) = 0. \tag{3.3'}$$

Denote by u_L and u_R the two prescribed states at $-\infty$ and $+\infty$ which are supposed to be connected by a solution of (3.3'). These states have to satisfy the Rankine-Hugoniot condition (1.4) which, for a stationary shock, is

$$f(u_L) = f(u_R). \tag{3.4}$$

In addition we suppose that the entropy condition is satisfied, i.e., that the shock speed is less than $-A(u_L)$, the sound speed to the left of the shock, and greater than $-A(u_R)$, the sound speed to the right:

$$A(u_L) < 0 < A(u_R). \tag{3.5}$$

It follows from (3.5) that there exists a value u_M such that

$$A(u_M) = 0. \tag{3.6}$$

We claim now that the following lattice function is a solution of (3.3'):

$$u(x) = \begin{cases} u_L & \text{for all negative lattice points,} \\ u_M & \text{for } x = 0, \\ u_R & \text{for all positive lattice points.} \end{cases} \tag{3.7}$$

We have to verify that the difference equation (3.3') is satisfied for all triplets of three successive lattice points. Since (3.3') is trivially satisfied whenever $a = b = c$, only three cases remain to be checked:

I	II	III
$a = u_L$	u_L	u_M
$b = u_L$	u_M	u_R
$c = u_M$	u_R	u_R

In case I, it follows from the first half of inequality (3.5) and from (3.6) that $\text{Max}\left\{A(b), A(c)\right\} = 0$; since $b-a$ is likewise zero, (3.3') is satisfied. We can show similarly that (3.3') is satisfied in case II.

In case II, the left side of (3.3') is

$$A(u_R)\,[u_R - u_M] + A(u_L)\,[u_M - u_L].$$

Since $A(u_M) = 0$, this expression is twice

$$\frac{A(u_M) + A(u_R)}{2} [u_R - u_M] + \frac{A(u_L) + A(u_M)}{2} [u_M - u_L].$$

We rewrite the two terms above as

$$[f(u_R) - f(u_M)] + [f(u_M) - f(u_L)]$$

which is correct if f is a quadratic function and which for general nonlinear f involves the same kind of error which we have already committed. This last expression is equal to $f(u_R) - f(u_L)$, a quantity equal to zero by virtue of the Rankine-Hugoniot relation (3.4). This verifies the difference equation (3.3") in case II.

We summarize our result in this

Theorem: Any two states u_L and u_R which can be connected by a stationary shock, i.e., which satisfy the Rankine-Hugoniot relation (3.4) and the entropy condition (3.5), can be connected by a steady state solution of our difference equation (3.2). This steady state solution is given approximately by formula (3.7), which shows that the transition region is spread over two mesh widths.

The theorem above refers to the case when B is taken to be 1. It is not so easy to solve the steady state difference equation for any other value of B, but we conjecture that a steady state solution exists for a reasonable range of B. On the basis of previous experience of

other investigators with artificial viscosity one would expect the width of the region across which the bulk of the transition takes place to increase with increasing B. This prediction was indeed borne out by numerical experiments recorded in the tables at the end of this paper.

Likewise, when u_L and u_R are two states which can be connected by a shock proceeding at some nonzero speed, we expect that the difference equation (2.6), with Q being given by (3.1), possesses a steady, progressing solution connecting these two states, whose structure is similar to the stationary solution.

We shall analyse now the stability of the difference scheme (2.6), with Q given by (3.1). We make the assumption that the stability of (2.6) is governed by the stability of the following associated linear difference operator:

$$I + \lambda A \Delta' + \frac{1}{2} \lambda^2 A^2 \Delta^2 + \frac{1}{2} \lambda Q \Delta^2, \tag{3.8}$$

where A and Q denote values of A and Q at some point. The amplification factor associated with this operator is

$$1 + i\lambda A \sin \alpha + \frac{1}{2} \left\{ \lambda^2 A^2 + \lambda Q \right\} \left\{ \cos \alpha - 1 \right\}. \tag{3.9}$$

It is not hard to show, by an analysis parallel to that given at the end of section 2, that this factor does not exceed 1 in absolute value for any frequency α if and only if

$$\lambda^2 A^2 + \lambda Q \leq 1. \tag{3.10}$$

Denote by μ the largest possible value of $|A|$; it follows from Eq. (3.1) for Q that then Q will not exceed $\frac{1}{2}BV$, where V is the variation of A. If A does not change sign, $V \leq \mu$. Substituting the above bounds into (3.10), we get

$$\lambda^2 \mu^2 + \frac{1}{2} B \lambda \mu \leq 1, \tag{3.11}$$

which is equivalent to

$$\lambda\mu \leq (1 + B^2/16)^{1/2} - B/4. \tag{3.11'}$$

In particular, we get for $B = 1$: $\frac{\sqrt{17}-1}{4} \geq \lambda\mu$

$$\lambda\mu \leq .78 .$$

This stability condition is slightly more stringent than the Courant-Friedrichs-Lewy condition (2.7). It bears a strong resemblance to the stability condition derived by von Neumann and Richtmyer in [13], and to the stability criterion derived later by George White of LASL.

If in (3.10) the strict inequality holds, then the amplification factor (3.9) is actually less than 1 in absolute value for all frequencies

α except $\alpha = 2\pi n$, n integer. This would lead us to believe that then the steady state solutions are strongly stable, in the following sense:

Every solution of the difference equation (2.6), with Q given by (3.1), whose initial values tend to u_L and u_R as x tends to $\pm\infty$, approaches a steady state solution with increasing t, provided that the stability condition (3.11') is satisfied.

Given an arbitrary initial state, such that the corresponding exact solution of the differential equation (1.1) contains a number of shocks, not necessarily stationary or steady, progressing, we would nevertheless expect the corresponding solution of the difference equation (2.6), with the same initial data, to bridge these shocks by transitions similar to the stationary solution we have found before, i.e., we expect the bulk of the transition to be spread over 2-3 mesh widths. This is reasonable, since the rate of variation of the states at the two sides of the shock will in general be much slower than the rapid variation within the shock itself.

Test calculations performed so far have confirmed this expectation. As yet none of these calculations have included very rapidly changing shocks, so at this time we don't know how the present method will handle such a situation.

4. ARTIFICIAL VISCOSITY FOR SYSTEMS OF CONSERVATION LAWS

In case of a system of any number of conservation laws the three properties of Q listed at the beginning of section 3 no longer suffice to determine Q within a single dimensionless constant but still leave a rather bewildering variety of possibilities.

Richtmyer and von Neumann based their choice of the artificial viscosity on a physical analogy, and the various ingenious modifications proposed and tested by Landshoff, Harlow and Longley were likewise partly based on physical analogies. In this section we propose a form for Q which is dictated entirely by mathematical analysis.

Again we study the shape of stationary solutions of (2.6):

$$\Delta' f + \frac{1}{2} \lambda \Delta A^2 \Delta + \frac{1}{2} \Delta Q \Delta u = 0. \tag{4.1}$$

We denote again by a, b and c three consecutive values of u, and as before make the following approximation:

$$\begin{aligned} 2\Delta' f &= f(c) - f(a) = f(c) - f(b) + f(b) - f(a) \\ &\approx A(b,c)(c-b) + A(a,b)(b-a), \end{aligned} \tag{4.2}$$

where as before $A(u,v)$ abbreviates $\frac{A(u) + A(v)}{2}$. Substituting this approximation into (4.1), we transform it to the following form:

$$\left\{A + \lambda A^2 + Q\right\} (c - b) + \left\{A - \lambda A^2 - Q\right\} (b - a) = 0, \qquad (4.3)$$

where A and Q in the first brace are to be evaluated between the points b and c, in the second brace between the points a and b. We wish to reduce the above difference equations for a vector valued function to a scalar equation. Such a reduction is rigorously possible if all the coefficient matrices commute; in that case we can get n difference equations similar to (3.3) where the role of A and Q is played by the eigenvalues of A and Q. This dictates the following choice for Q: $Q(a,b)$ should be a matrix commuting with $A(a,b)$ whose eigenvalues are equal to the absolute values of the differences of the corresponding eigenvalue of $A(u)$ and $A(v)$ times dimensionless factors $B_1, \ldots, B_n$ of the order of magnitude 1.

The requirement that Q should commute with A implies, according to a well known theorem of matrix theory, that Q is a function of A. This function we can take to be a polynomial of degree $n - 1$:

$$Q = g_o I + g_1 A + \cdots + g_{n-1} A^{n-1} \qquad (4.4)$$

whose coefficients $g_o, \ldots, g_{n-1}$ are uniquely determined by the proposed choice for the eigenvalues of Q. Finding the coefficients

$g_0, \ldots, g_{n-1}$ leads to the Lagrange interpolation problem which is easily solved.

In the next section we shall actually carry out the determination of Q in case of the equations of motion of a compressible fluid using Lagrange coordinates.

It should be pointed out that even if $Q(u,v)$ is chosen to commute with $A(a,b)$, there is an error involved in replacing (4.3) by scalar equations since the matrices $A(a,b)$ and $A(b,c)$ do <u>not</u> commute in general. Still we feel that the above choice for Q comes pretty close to imitating the scalar case.

We close this section by a further observation on the shape of the solutions of the steady state equation (4.1). In analysing this equation in the scalar case we have simplified matters by dropping the term $\frac{1}{2}\lambda A^2$, which was thought to be small compared to the artificial viscosity term in the region of rapid variation. On the other hand, in determining the shape of the tail ends of the transition curve the role of the two terms is reversed. We shall determine now asymptotically the shape of the tail ends. We write

$$a = u + r^k w,$$

$$b = u + r^{k+1} w,$$

$$c = u + r^{k+2} w,$$

where u is one of the two values u_L or u_R and w some vector independent of k. We substitute this into (4.3), divide by r^k and drop all terms that still contain the factor r^k. In particular the quadratic term Q drops out. We get

$$\left\{A + \lambda A^2\right\} (r^2 - r) + \left\{A - \lambda A^2\right\} (r - 1)\, w = 0,$$

where A denotes $A(u)$. This equation has a nontrivial solution if and only if the matrix acting on w has zero as eigenvalue. Now according to the spectral mapping theorem, the eigenvalues of that matrix are

$$\left\{\mu + \lambda \mu^2\right\} (r^2 - r) + \left\{\mu - \lambda \mu^2\right\} (r - 1),$$

where μ stands for an eigenvalue of A. Setting the above expression equal to zero gives, after eliminating the uninteresting root $r = 1$, the following value for r:

$$r = \frac{\lambda\mu - 1}{\lambda\mu + 1}.$$

According to the Courant-Fredrichs-Lewy condition (2.7), $|\lambda\mu| \leq 1$, so that the above expression is always negative. Furthermore, at the left endpoint u_L we must have $|r| > 1$, at the right end u_R we must have $|r| < 1$. This is the case if and only if $\mu(u_L)$ is negative, $\mu(u_R)$ is

positive. But according to the entropy condition described at the beginning of section 1, two states u_L and u_R which can be connected by a stationary shock always possess such eigenvalues.

A similar analysis can be made of the asymptotic shape of the tail of steady, progressing solutions of the difference equation (2.6). We are led to an equation which contains noninteger powers of r and which may have complete solutions.

All this is not very important as a practical consideration, but it does explain a curious phenomenon observed already in calculations performed with the Richtmyer-von Neumann method, and in many calculations using modifications of that method: that the shock transition overshoots and approaches a constant value in an oscillatory fashion. By changing the available parameter, the amount of overshooting could be diminished but never completely eliminated, nor could the oscillatory nature of the approach to a constant value be changed. The present analysis explains this behaviour by the negative value of r.

In contrast, calculations performed by a somewhat crude earlier method proposed by Lax (see [6] and [7]) produced shock transitions which approach their final values monotonically. A similar asymptotic study of the tail end of the steady state solutions for these equations disclosed that all the relevant values of v are positive.

5. APPLICATION TO HYDRODYNAMICS

In section 4 we have sketched a method for constructing an artificial viscosity term for use in a numerical scheme for arbitrary systems of conservation laws. In this section we shall carry out the details of this construction in the special case of the equations of compressible flow in Lagrange coordinates.

As dependent variables we shall use specific[3] volume, momentum and total energy, denoted by V, v and E. The quantity v is of course velocity, and the total energy E is the sum of internal and kinetic energy:

$$E = e + \frac{1}{2} v^2. \tag{5.1}$$

The internal energy e is related to pressure p and specific volume V by the equation of state

$$p = p(e,V).$$

[3] I.e., per unit mass.

The equations of conservation of mass, momentum and energy are:

$$\begin{aligned} V_t &= v_x, \\ v_t &= -p_x, \\ E_t &= -(vp)_x. \end{aligned} \tag{5.2}$$

Here t is time and x is Lagrange mass variable. Differentiating these equations with respect to time, we obtain

$$\begin{aligned} V_{tt} &= v_{tx}, \\ v_{tt} &= -p_{tx}, \\ E_t &= (vp)_{tx}. \end{aligned} \tag{5.3}$$

Using the chain rule, we have

$$p_t = p_e e_t + p_v V_t. \tag{5.4}$$

According to a well known identity in thermodynamics

$$ppe - p_V = c^2, \tag{5.5}$$

where C is the Lagrangian sound speed. We can also write

$$e_t = E_t - v\, v_t. \tag{5.6}$$

Using (5.4), (5.5), (5.6) and the original differential equations (5.2) to express first t derivatives as x derivatives, we can rewrite (5.3) as follows:

$$\begin{aligned} V_{tt} &= -p_{xx}, \\ v_{tt} &= (C^2 v_x)_x, \\ E_{tt} &= (pp_x + C^2 vv_x)_x. \end{aligned} \tag{5.7}$$

Recalling the identity valid for arbitrary systems of conservation laws;

$$u_{tt} = (A^2 u_x)_x,$$

we see from (5.7) that in our case

$$A^2 u_x = \begin{pmatrix} -p_x \\ C^2 v_x \\ pp_x + C^2 vv_x \end{pmatrix}. \tag{5.8}$$

Accordingly we shall use the approximation

$$A^2 \Delta u = \begin{pmatrix} -\Delta p \\ C^2 \Delta v \\ p \Delta p + C^2 v \Delta v. \end{pmatrix}. \tag{5.9}$$

We turn now to the determination of Q, which, according to the recipe given in section 4, is a quadratic polynomial in $A(a,b)$ whose eigenvalues are constant multiples of the absolute values of the differences of the eigenvalues of $A(a)$ and $A(b)$. Now the eigenvalues of A are: $0, \pm C$. We claim that Q is given by

$$Q = \frac{B}{2} \frac{|C(a) - C(b)|}{C^2} A^2;$$

A and C in the above formula are to be evaluated at $\frac{a+b}{2}$. Clearly, Q as given by the above formula has the appropriate eigenvalues.

Substituting this form of Q into formulas (2.4) (2.5) for g and making use of (5.9), we have

$$g(a,b) = \begin{pmatrix} \bar{v} - \frac{1}{2}\left\{B \frac{|\Delta C|}{2\bar{C}^2} + \lambda\right\} \Delta p \\ -\bar{p} + \frac{1}{2}\left\{\frac{B|\Delta C|}{2\bar{C}^2} + \lambda\right\} \bar{C}^2 \Delta v \\ -\overline{vp} + \frac{1}{2}\left\{B \frac{|\Delta C|}{2\bar{C}^2} + \lambda\right\}\left\{\bar{p}\Delta p + \bar{C}^2 \bar{v}\Delta v\right\} \end{pmatrix}, \tag{5.10}$$

where the barred quantities are to be evaluated as averages between points a and b.

Two observations about this formula are in order: first,

1. Consider an initial distribution in which v and p are constant, although V not necessarily; in fact, V may be discontinuous. For such initial values the function g, given by formula (5.10), is a constant; therefore the corresponding solution of the difference equation (5.2) is stationary. This agrees, of course, with the fact that such an initial configuration is--in the absence of diffusion or heat conduction--in hydrodynamic equilibrium and shows that our difference scheme introduces no artificial diffusion or heat conduction.

More generally, we expect that even in nonstationary flows contact discontinuities are transmitted as sharp discontinuities.

2. Although our formula (5.10) for g was derived on a purely mathematical basis, it can be given a more intuitive interpretation. First of all, as already observed in section 2, the second order correction terms can be regarded as merely centering the values of f properly. The additional artificial viscosity term can be given the following interpretation: Define an artificial velocity and an artificial viscous pressure as follows:

$$v_{art} = -\frac{B\Delta x^2}{4c^2} |c_x| p_x,$$

$$p_{art} = -\frac{B\Delta x^2}{4} |c_x| v_x,$$

where the indicated derivatives are to be replaced by centered difference quotients. Clearly, the effect of the Q term in the mass and momentum equations is to augment the value of velocity and pressure--properly centered--by the amounts indicated above. The effect of the Q term on the energy equation can be described similarly, if we neglect the difference between $\overline{vp}$ and $\bar{v}\,\bar{p}$, and if we neglect the product of the artificial velocity and the artificial viscous pressure.

The artificial viscous pressure that has cropped up in this treatment bears a strong resemblance to the one introduced by Rolf Landshoff.

We present the results of two calculations using (1.6) and (5.10) to obtain approximate solutions of (5.2). In the first calculation we initially had two constant states separated by a shock. The exact shock speed is 1. In Table I we show the appearance of the configuration at the 40th time cycle, which, with $\frac{\Delta t}{\Delta x} = .337$, corresponds to $t = 13.5$. At this time the shock, which started at $x = 50$, should be at $x = 63.5$. We used 2 for the parameter B, and the second order correction terms were not switched off.

In the second calculation we again initially have two constant states separated by a discontinuity. However, this time the configuration at 40 cycles is a shock moving with speed 1.24, a contact discontinuity at the point of the initial discontinuity, namely $x = 50$, and a rarefaction wave. We used $\frac{\Delta t}{\Delta x} = .337$, $B = 1$. The results of the calculation are listed in Table II.

TABLE I

PROGRESSING SHOCK

x	VOLUME	VELOCITY	ENERGY	
44	1.000	1.000	4.429	
45	1.000	1.000	4.429	
46	1.000	1.000	4.429	
47	1.000	1.000	4.429	
48	1.000	1.000	4.429	
49	1.000	1.000	4.428	
50	1.001	1.000	4.434	← Initial Position of Shock
51	1.044	1.000	4.601	
52	1.012	1.000	4.476	
53	1.003	1.001	4.445	
54	1.001	1.001	4.436	
55	1.002	.998	4.426	
56	1.002	.998	4.425	
57	.996	1.006	4.442	
58	.995	1.007	4.444	
59	1.010	.986	4.401	
60	1.014	.980	4.388	
61	.979	1.032	4.496	
62	.965	1.054	4.545	
63	1.103	.864	4.172	← Present Position of Shock
64	1.503	.411	3.384	
65	1.876	.085	2.944	
66	1.983	.011	2.867	
67	1.998	.001	2.858	
68	2.000	.000	2.857	
69	2.000	.000	2.857	
70	2.000	.000	2.857	

TABLE II

PROGRESSING SHOCK, STATIONARY CONTACT DISCONTINUITY, AND RAREFACTION WAVE

x	VOLUME	VELOCITY	ENERGY	
24	2.245	.698	20.04	
25	2.246	.699	20.04	p = 3.528
26	2.248	.702	20.03	
27	2.253	.709	20.02	
28	2.264	.725	20.00	
29	2.284	.754	19.95	
30	2.316	.800	19.87	
31	2.362	.866	19.78	
32	2.422	.948	19.66	
33	2.495	1.045	19.53	Rarefaction Wave
34	2.577	1.150	19.40	
35	2.660	1.259	19.28	
36	2.756	1.366	19.18	
37	2.842	1.463	19.09	
38	2.912	1.541	19.03	Exact Volume = 2.900
39	2.956	1.589	19.00	
40	2.963	1.596	19.00	
41	2.935	1.566	19.02	
42	2.898	1.525	19.05	
43	2.882	1.508	19.06	
44	2.892	1.518	19.05	
45	2.907	1.534	19.04	
46	2.909	1.536	19.04	
47	2.903	1.529	19.05	
48	2.902	1.526	19.06	p = 2.465
49	2.907	1.528	19.08	
50	2.887	1.528	18.96	← Initial Discontinuity
51	.825	1.528	6.253	
52	.777	1.528	5.956	
53	.772	1.528	5.930	
54	.774	1.528	5.937	
55	.773	1.528	5.932	Exact Volume = .767
56	.772	1.528	5.924	Exact Velocity = 1.528
57	.771	1.527	5.914	
58	.769	1.528	5.906	
59	.768	1.527	5.900	
60	.767	1.527	5.895	
61	.766	1.528	5.894	
62	.765	1.533	5.903	
63	.766	1.526	5.889	
64	.770	1.519	5.870	
65	.749	1.576	6.011	
66	.754	1.546	5.983	
				Present Exact
67	1.204	.850	4.309	← Position of Shock
68	1.852	.108	2.979	
69	1.990	.006	2.863	
70	2.000	.000	2.857	p = .5714

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